Computing the permanental polynomials of bipartite graphs by Pfaffian orientation[*]

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Abstract

The permanental polynomial of a graph G is $\pi(G, x) \triangleq \mathrm{per}(xI - A(G))$. From the result that a bipartite graph G admits an orientation G^e such that every cycle is oddly oriented if and only if it contains no even subdivision of $K_{2,3}$, Yan and Zhang showed that the permanental polynomial of such a bipartite graph G can be expressed as the characteristic polynomial of the skew adjacency matrix $A(G^e)$. In this paper we first prove that this equality holds only if the bipartite graph G contains no even subdivision of $K_{2,3}$. Then we prove that such bipartite graphs are planar. Further we mainly show that a 2-connected bipartite graph contains no even subdivision of $K_{2,3}$ if and only if it is planar 1-cycle resonant. This implies that each cycle is oddly oriented in any Pfaffian orientation of a 2-connected bipartite graph containing no even subdivision of $K_{2,3}$. As applications, permanental polynomials for some types of bipartite graphs are computed.

Key Words: Permanent; Permanental polynomial; Determinant; 1-cycle resonant; Pfaffian orientation.

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1 Introduction

In this article, we always consider finite and simple graphs. Let G be a graph with vertex-set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge-set $E(G) = \{e_1, e_2, \cdots, e_m\}$. The adjacency matrix $A(G) = (a_{ij})_{n \times n}$ of G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is adjacent to vertex } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

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The permanent of an $n \times n$ matrix A is defined as

$$\mathrm{per}(\mathrm{A}) = \sum_{\sigma \in \Lambda_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1}$$

where Λ_n denotes the set of all permutations of $\{1, 2, \cdots, n\}$. The permanental polynomial of G is defined as

$$\pi(G, x) = \mathrm{per}(xI - A(G)) = \sum_{k=0}^{n} b_k x^{n-k}, \tag{2}$$

where I is the identity matrix of order n. It is easy to see that $(-1)^k b_k$ is the sum of the $k \times k$ principle subpermanents of A [17]. It was mentioned in [2, 17] that if G is a bipartite graph, then $b_{2k} \geq 0$ and $b_{2k+1} = 0$ for all $k \geq 0$; In particular,

$$b_n = \mathrm{per}(A(G)) = m^2(G), \tag{3}$$

where $m(G)$ is the number of perfect matchings of G.

It was in 1981 that the permanental polynomial was firstly investigated in chemistry in [14], where some relations between the permanental polynomial and the structure of conjugated molecules were discussed. Later, Cash [3, 4] investigated the mathematical properties of the coefficients and zeros of the permanental polynomials of some chemical graphs. This suggests that the permanental polynomial encodes a variety of structural information. Gutman and Cash also demonstrated [12] several relations between the coefficients of the permanental and characteristic polynomials. See [5, 7, 13] for more about the permanental polynomials. We can compute the determinant of an $n \times n$ matrix efficiently by Gaussian elimination. Although $\mathrm{per}(A)$ looks similar to $\det(A)$, it is harder to be computed. Valiant proved [19] that it is a #P-complete problem to evaluate the permanent of a $(0, 1)$-matrix. For these reasons, we want to compute the permanental polynomials of graphs by the determinant of a matrix.

Pfaffian orientations of graphs can be used to the enumeration of perfect matchings. Now we recall some definitions. A subgraph H of a graph G is called *nice* if $G - V(H)$ has a perfect matching. Let G^e be an orientation of a graph G and C a cycle of even length in G. C is *oddly oriented* in G^e if C contains an odd number of edges that are directed in either direction of the cycle. An orientation of G is *Pfaffian* if every nice cycle C is oddly oriented. The skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows,

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G^e), \\ -1 & \text{if } (v_j, v_i) \in E(G^e), \\ 0 & \text{otherwise.} \end{cases}$$

If a bipartite graph G has a Pfaffian orientation G^e, then [16]

$$\mathrm{per}(A(G)) = \det(A(G^e)) = m^2(G). \tag{4}$$

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We say an edge e of a graph G is *oddly subdivided* in a graph G' if G' is obtained from G by subdividing e an odd number of times. Otherwise, we say e is evenly subdivided. The graph G' is said to be *an even subdivision of a graph G* if G' can be obtained from G by subdividing evenly each edge of G.

Fisher and Little [8] gave a characterization of a graph that has an orientation with each cycle being oddly oriented. From this result, we have that a bipartite graph G admits an orientation G^e such that every cycle is oddly oriented if and only if it contains no even subdivision of $K_{2,3}$. Accordingly, Yan and Zhang showed [21] that the permanental polynomial of such a bipartite graph G can be expressed as the characteristic polynomial of the skew adjacency matrix $A(G^e)$: $\pi(G, x) = \det(xI - A(G^e))$.

In this paper we want to investigate bipartite graphs for which the permanental polynomials can be computed by the characteristic polynomials. Following Yan and Zhang's result, in Section 2 we obtain that a bipartite graph admits this property if and only if it has a Pfaffian orientation such that all the cycles are oddly oriented, if and only if it contains no even subdivision of $K_{2,3}$.

In Section 3 we mainly characterize bipartite graphs containing no even subdivision of $K_{2,3}$. The starting point of this section is to show that these graphs are planar. Then we find that if such graphs are 2-connected, they each has a bipartite ear decomposition starting with any cycle. This implies that such graphs are elementary bipartite graphs. Unexpectedly, we obtain the main result that a 2-connected bipartite graph G contains no even subdivision of $K_{2,3}$ if and only if it is planar 1-cycle resonant. In fact, planar 1-cycle resonant graphs have already been introduced and investigated extensively in [9, 10, 11, 20]. Various characterizations [10] and constructional features [11] for planar 1-cycle resonant graphs have been given. These characterizations enable ones to design efficient algorithms to decide whether a 2-connected planar bipartite graph is 1-cycle resonant [11, 20].

From the previous main result, we have that each cycle of a 2-connected bipartite graph G containing no even subdivision of $K_{2,3}$ is nice and is oddly oriented in any Pfaffian orientation of G. Finally an algorithm [16] is recalled to give a Pfaffian orientation of a plane graph. As applications, the permanental polynomials of some types of bipartite graphs are computed. In particular, we obtain explicit expressions for permanental polynomials of two classes of graphs G_1^s and G_2^r.

2   A criterion for computing permanental polynomials

An elegant characterization for Pfaffian bipartite graphs was given by Little.

Theorem 2.1. *[15] A bipartite graph admits a Pfaffian orientation if and only if it does not contain an even subdivision of $K_{3,3}$ as a nice subgraph.*

Fisher and Little gave a characterization for the existence of an orientation of a graph

such that all the even cycles are oddly oriented as follows.

Theorem 2.2. *[8] A graph has an orientation under which every cycle of even length is oddly oriented if and only if the graph contains no subgraph which is, after the contraction of at most one cycle of odd length, an even subdivision of $K_{2,3}$.*

For bipartite graphs, we have the following immediate corollary.

Corollary 2.3. *There exists an orientation of a bipartite graph G such that all the cycles of G are oddly oriented if and only if G contains no even subdivision of $K_{2,3}$.*

Based on such results, Yan and Zhang found that the permanental polynomial of a bipartite graph that has no even subdivision of $K_{2,3}$ can be computed by the determinant.

Theorem 2.4. *[21] Let G be a bipartite graph containing no even subdivision of $K_{2,3}$. Then there exists an orientation G^e of G such that*

$$\pi(G, x) = \det(xI - A(G^e)). \tag{5}$$

In fact we can show that the converse of the theorem is also valid.

Theorem 2.5. *Let G be a bipartite graph of order n. Then an orientation G^e of G satisfies $\pi(G, x) = \det(xI - A(G^e))$ if and only if each cycle of G is oddly oriented in G^e.*

Proof. Let

$$\det(xI - A(G^e)) = \sum_{i=0}^{n} a_i x^{n-i}.$$

Then we have

$$a_{2k+1} = b_{2k+1} = 0, k = 0, 1, 2, ...,$$

and

$$b_{2k} = \sum_{H} \text{per}(A(H)), a_{2k} = \sum_{H} \det(A(H^e)), k = 0, 1, ...,$$

where both sums range over all induced subgraphs H of G with $2k$ vertices.

For the complete we reprove the sufficiency. If G has an orientation G^e such that every cycle is oddly oriented, then the restriction of G^e on each induced subgraph H of G is a Pfaffian orientation H^e. Hence $\text{per}(A(H)) = \det(A(H^e))$. That means that $b_i = a_i$ for each $i \geq 1$. Hence $\pi(G, x) = \det(xI - A(G^e))$.

We now prove the necessity. Suppose that G has a cycle C that is evenly oriented in G^e. For convenience, let $V(C) = \{1, 2, \ldots, 2k\}$ and $C = 12 \cdots (2k)1$. Now we consider the subgraph $G[C]$ induced by the vertices of C. Let $G^e[C]$ be the orientation G^e restricted on $G[C]$. From the definitions of permanents and determinants, we have that $\text{per}(A(G[C])) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{2k\sigma(2k)}$ and $\det(A(G^e[C])) = \sum_{\sigma} \text{sgn}(\sigma) a'_{1\sigma(1)} a'_{2\sigma(2)} \cdots a'_{2k\sigma(2k)}$, where $a_{i\sigma(i)}$ and $a'_{i\sigma(i)}$ are the elements in the i-th row and $\sigma(i)$-th column of $A(G[C])$ and $A(G^e[C])$,

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respectively, and both sums range over all the permutations of $\{1, 2, \cdots, 2k\}$. For any given σ, if some $a_{i\sigma(i)}$ equals zero, then $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{2k\sigma(2k)} = a'_{1\sigma(1)}a'_{2\sigma(2)}\cdots a'_{2k\sigma(2k)} = 0$. Otherwise, $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{2k\sigma(2k)} = 1$ and $a'_{1\sigma(1)}a'_{2\sigma(2)}\cdots a'_{2k\sigma(2k)} = 1$ or -1. Hence $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{2k\sigma(2k)} \geq \text{sgn}(\sigma)a'_{1\sigma(1)}a'_{2\sigma(2)}\cdots a'_{2k\sigma(2k)}$. For the cycle $\sigma = (12\cdots 2k)$, we know that $\text{sgn}(\sigma) = -1$ and $a'_{1\sigma(1)}a'_{2\sigma(2)}\cdots a'_{2k\sigma(2k)} = 1$ since C is evenly oriented. So $\text{sgn}(\sigma)a'_{1\sigma(1)}a'_{2\sigma(2)}\cdots a'_{2k\sigma(2k)} = -1$, while $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{2k\sigma(2k)} = 1$. Hence $\det(A(G^e[C])) < \text{per}(A(G[C]))$.

In an analogous argument, we have that $\det(A(G^e[H])) \leq \text{per}(A(G[H]))$ for any induced subgraph H of G. Hence $a_{2k} = \sum_{H} \det(A(H^e)) < \sum_{H} \text{per}(A(H)) = b_{2k}$ and $\pi(G, x) \neq \det(xI - A(H^e))$. $\qquad\square$

Corollary 2.6. *There exists an orientation G^e of a bipartite graph G such that $\pi(G, x) = \det(xI - A(G^e))$ if and only if G contains no even subdivision of $K_{2,3}$.*

Proof. It follows immediately from Theorem 2.5 and Corollary 2.3. $\qquad\square$

3 Characterizations and recognition of bipartite graphs containing no even subdivision of $K_{2,3}$

In this section we will characterize bipartite graphs containing no even subdivision of $K_{2,3}$. We first show that such graphs are planar. Then we show that a 2-connected bipartite graph has no even subdivision of $K_{2,3}$ if and only if it is a planar 1-cycle resonant graph.

Theorem 3.1. *[1, Kuratowski's theorem] A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

Lemma 3.2. *If a bipartite graph contains no even subdivision of $K_{2,3}$, then it is planar.*

Proof. We prove the converse-negative proposition. Suppose that a bipartite graph G is not planar. Then by Theorem 3.1 it contains a subdivision H of $K_{3,3}$ or K_5.

If an edge e of $K_{3,3}$ or K_5 is oddly subdivided in H, then P is the path of even length in H obtained by subdividing e. There are two cycles C_1, C_2 containing e in $K_{3,3}$ or K_5 with $C_1 \cap C_2 = \{e\}$, and the corresponding cycles in H are denoted by C'_1, C'_2. Since H is bipartite, in $C'_1 \cup C'_2$ there are three pairwise internally disjoint paths of even length connecting the two endvertices of P. So we obtain an even subdivision of $K_{2,3}$.

If H is an even subdivision of $K_{3,3}$ or K_5, then $K_{3,3}$ or K_5 always contains $K_{2,3}$ as a subgraph, which corresponds to an even subdivision of $K_{2,3}$ in H. $\qquad\square$

A sequence of subgraphs of G, $(G_0, G_1, \cdots, G_m)$ is a *bipartite ear decomposition* of G if G_0 is an edge x, $G_m = G$, every G_i for $i = 1, 2, \cdots, m$ is obtained from G_{i-1} by adding an path P_{i-1} of odd length which is openly disjoint from G_{i-1} but its endvertices belong to G_{i-1}. Such an ear decomposition can also be denoted as $G := x + P_0 + P_1 + \cdots + P_{m-1}$.

Lemma 3.3. *The 2-connected bipartite graph G containing no even subdivision of $K_{2,3}$ has a bipartite ear decomposition starting with any cycle in it.*

Proof. Let G_1 be any cycle in G. We want to show that G has a bipartite ear decomposition starting with G_1 by induction. Let G_i be a subgraph of G obtained by successively adding $i - 1$ ears, $i \geq 1$. If $G_i \neq G$, then an edge uv exists in $G - E(G_i)$. For an edge $xy \in E(G_i)$, uv and xy lie on a common cycle C since G is 2-connected. Let P be a path in C such that the intersections of P and G_i are the both endvertices a and b of P. If a and b have the same color, then P is a path of even length. Since G_i is 2-connected, G_i has two internally disjoint paths of even length connecting a and b. Hence three pairwise internally disjoint paths of even length of G connect a and b. That is, an even subdivision of $K_{2,3}$ exists in G and a contradiction occurs. So a and b have different colors and P is a path of odd length. Now $G_{i+1} := G_i + P$ is a subgraph of G obtained by successively adding i ears, $i \geq 1$. Hence G has a bipartite ear decomposition starting with any cycle in it. □

The above result shows that such graphs relate with 1-cycle resonant graphs. A connected graph is said to be *k-cycle resonant* if, for $1 \leq t \leq k$, any t disjoint cycles in G are mutually resonant, that is, there is a perfect matching M of G such that each of the t cycles is an M-alternating cycle. k-cycle resonant graphs were introduced by Guo and Zhang [9] as a natural generalization of k-resonant benzenoid systems, which originate from Clar's aromatic sextet theory and Randić's conjugated circuit model. They obtained that a k-cycle resonant graph is bipartite [9]. In the following theorem, we can see that such two types of graphs are equivalent under the 2-connected condition.

Theorem 3.4. *A 2-connected bipartite graph G contains no even subdivision of $K_{2,3}$ if and only if G is planar 1-cycle resonant.*

Proof. We first prove the necessity. From Lemma 3.3, G has a bipartite ear decomposition $G := C + P_1 + P_2 + \cdots + P_r$ starting with any cycle C in it. Because the ears P_i's for $1 \leq i \leq r$ are all of odd length, the the graph $G - V(C)$ has a perfect matching M which covers all the internal vertices of each P_i. So every cycle of G is nice. By Lemma 3.2 G is planar. Hence G is planar 1-cycle resonant.

Now we prove the sufficiency. If G contains a subgraph H which is an even subdivision of $K_{2,3}$, then in the subgraph H, there are three pairwise internally disjoint paths l_1, l_2 and l_3 of even length joining two given vertices. Since G is planar, we imbed it in the plane so that l_2 lies in the interior of the cycle $C := l_1 \cup l_3$. Since G is 1-cycle resonant, $C_1 := l_1 \cup l_2$ and $C_2 := l_2 \cup l_3$ are nice cycles of G. Hence there is an even number of vertices in the interior of C_1 and C_2 respectively. Further, since l_2 has an odd number of internal vertices, there is an odd number of vertices in the interior of C. This implies that C is not a nice cycle of G, contradicting that G is 1-cycle resonant. □

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A connected bipartite graph G is *elementary* if each edge is contained in a perfect matching of G. For more details about such graphs see [22]. Then by Lemmas 3.2 and 3.3 or Theorem 3.4 we have the following result.

Corollary 3.5. *The 2-connected bipartite graph G containing no even subdivision of $K_{2,3}$ is a planar and elementary bipartite graph.*

A *block* of a connected graph G is a maximal connected subgraph of G without cutvertices. From Theorem 3.4 we have the following general result.

Corollary 3.6. *A connected bipartite graph G contains no even subdivision of $K_{2,3}$ if and only if each block of G is planar 1-cycle resonant.*

From the proof of Theorem 3.4, we have the following result.

Corollary 3.7. *If a connected graph is a planar 1-cycle resonant graph, then it contains no even subdivision of $K_{2,3}$.*

We have seen in the previous theorem that 2-connected bipartite graphs containing no even subdivision of $K_{2,3}$ are equivalent to planar 1-cycle resonant graphs. In fact, various characterizations, the construction and recognition algorithms for planar 1-cycle resonant graphs have already been obtained in [9, 10, 11, 20]. Before stating these results, we need to give some terminology and notations.

Let H be a subgraph of a connected graph G. A *bridge* B of H is either an edge in $G - E(H)$ with two endvertices in H, or a subgraph of G induced by all edges incident with a vertex in a component B' of $G - V(H)$. An *attachment vertex* of a bridge B to H is a vertex in H which is incident with an edge in B. Two bridges of a cycle C *avoid* one another if all the attachment vertices of one bridge lie between two consecutive attachment vertices of the other bridge along C.

Following Theorem 3.4 and Theorem 1 of Ref. [10], we have the following characterizations.

Theorem 3.8. *Let G be a 2-connected bipartite planar graph. Then the following statements are equivalent:*

(1) G contains no even subdivision of $K_{2,3}$,

(2) G is planar 1-cycle resonant,

(3) For any cycle C in G, $G - V(C)$ has no odd component,

(4) For any cycle C in G, any bridge of C has exactly two attachment vertices which have different colors,

(5) For any cycle C in G, any two bridges of C avoid one another. Moreover, for any 2-connected subgraph B of G with exactly two attachment vertices, the attachment vertices of B have different colors.

The next result gives a structural description of planar 1-cycle resonant graphs.

Theorem 3.9. *[11] A 2-connected graph G is planar 1-cycle resonant graph if and only if G has a bipartite ear decomposition $G := C_0 + P_1 + \cdots + P_k$ such that C_0 is a cycle and each P_i $(1 \leq i \leq k)$ satisfies that (1) the endvertices x, y of P_i have different colors in $G_{i-1} = C_0 + P_1 + \cdots + P_{i-1}$, (2) either x and y are adjacent in G_{i-1} or $\{x, y\}$ is a vertex cut of G_{i-1}.*

This theorem can be used to construct bipartite graphs containing no even subdivision of $K_{2,3}$. In addition, it derives an algorithm of time complexity $O(n^2)$, where n is the number of vertices of G, to determine whether a 2-connected plane graph is 1-cycle resonant; See [11] for more details.

In [20] a linear-time algorithm with respect to the number of vertices to decide whether a 2-connected plane bipartite graph G is 1-cycle resonant was provided. This algorithm is designed by testing whether the attachment vertices of any bridge of the outer cycle C of G satisfy statement (4) in Theorem 3.8 and the attachment vertices u, v of any maximal 2-connected subgraph H of any bridge B of C have different colors. If the above conditions hold, we proceed recursively for $G := H$. Otherwise, G is not 1-cycle resonant.

If a given planar bipartite graph G is connected, we can implement the above method to each 2-connected block of G to test whether it is 1-cycle resonant. If answers are all "yes", the graph G contains no even subdivision of $K_{2,3}$. Hence we can present a linear time algorithm to determine whether a given planar bipartite graph G contains no even subdivision of $K_{2,3}$ in this approach.

By the way, we turn to outerplanar graphs. A graph is *outerplanar* if it has an embedding into the plane with every vertex on the boundary of the exterior face. The following characterizations for outerplanar graphs were given.

Theorem 3.10. *[6] A graph is outerplanar if and only if it contains no subdivision of $K_{2,3}$ or K_4.*

Theorem 3.11. *[18] A graph without triangles is outerplanar if and only if it contains no subdivision of $K_{2,3}$.*

For bipartite graphs, we have the following characterization by Theorem 3.11.

Corollary 3.12. *A bipartite graph is outerplanar if and only if it contains no subdivision of $K_{2,3}$.*

From Corollary 3.12 and the characterizations of a planar 1-cycle resonant graph, we can obtain the following result immediately.

Corollary 3.13. *Let $G := C + P_1 + \cdots + P_k$ be a bipartite ear decomposition obtained by Theorem 3.9. If every ear P_i is either a path of length 1 or joins two adjacent vertices of $C \cup P_1 \cup \cdots \cup P_{i-1}$ for $1 \leq i \leq k$, then G is outerplanar. If not, G contains a subdivision, but not an even subdivision, of $K_{2,3}$.*

4 Permanental polynomials of some graphs

In the last section some characterizations and recognition of bipartite graphs containing no even subdivision of $K_{2,3}$ are given. We now compute the permanental polynomials of such graphs. An algorithm is presented firstly to construct their orientations with each cycle being oddly oriented.

4.1 An orientation algorithm

The following algorithm has already been provided in [16] to construct Pfaffian orientations for plane graphs. Here we will show that, for a 2-connected bipartite graph G containing no even subdivision of $K_{2,3}$, each cycle of G is oddly oriented in any Pfaffian orientation of it.

Algorithm 4.1. *[16] Let G be a connected plane graph.*
1. Find a spanning tree T in G and orient it arbitrarily.
2. Let $G_1 = T$.
3. If $G_i = G$, stop. Otherwise, take an edge e_i of G not in G_i such that e_i and G_i bound an interior face f_i of G, and orient e_i such that an odd number of edges on the boundary of f_i is oriented clockwise.
4. Set $i + 1 = i$ and $G_{i+1} = G_i \cup \{e_i\}$. Go to step 3.

Theorem 4.2. *[16] Let G be a connected plane graph. The orientation G^e given by Algorithm 4.1 is a Pfaffian orientation of G. Such an orientation can be constructed in polynomial time.*

Theorem 4.3. *Let G be a 2-connected bipartite graph containing no even subdivision of $K_{2,3}$. Then each cycle is oddly oriented in a Pfaffian orientation G^e of it.*

Proof. Since G^e is a Pfaffian orientation of G, each nice cycle is oddly oriented. From Theorem 3.4, each cycle of G is nice. So each cycle of G is oddly oriented in G^e. □

For a connected bipartite graph containing no even subdivision of $K_{2,3}$, we can give an orientation so that each cycle is oddly oriented: For each 2-connected block of G, we implement Algorithm 4.1 to get its Pfaffian orientation; For each cut edge of G we orient it arbitrarily. By Theorem 4.3 we can see that for such an orientation of G each cycle is indeed oddly oriented.

Remark 4.4. *Implementing Algorithm 4.1 directly to a connected plane bipartite graph containing no even subdivision of $K_{2,3}$, we obtain a Pfaffian orientation, but cannot necessarily obtain the required orientation: an evenly oriented cycle may exist. See the graph G in Figure 1 for example. The subgraph induced by the solid edges is a spanning tree T with an orientation shown in Figure 1. By Algorithm 4.1, we first add the edge $v_5 v_6$ to T and orient it from v_5 to v_6 so that the interior face bounded by it is oddly oriented. Then we add the edge $v_3 v_4$ and orient it from v_4 to v_3 so that the face boundary $v_1 v_4 v_3 v_2 v_1 v_7 v_6 v_5 v_1$*
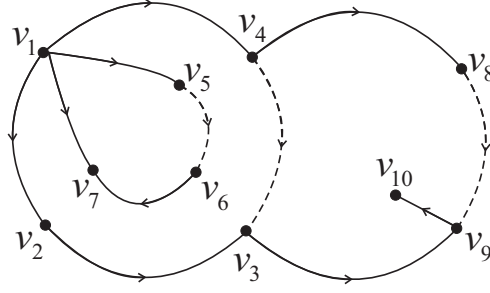
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Figure 1. The graph with evenly oriented cycles.

has an odd number of edges oriented clockwise along it. Finally we add the edge $v_8 v_9$ and orient it from v_8 to v_9 so that the face boundary $v_4 v_8 v_9 v_{10} v_9 v_3 v_4$ has an odd number of edges oriented clockwise along it. We can see that in the resulting Pfaffian orientation, the cycles $v_1 v_2 v_3 v_4 v_1$ and $v_3 v_4 v_8 v_9 v_3$ are evenly oriented.

4.2 A computational approach

Let $G = (U, V)$ be a bipartite graph with $|U| = |V|$. By choosing suitable ordering of vertices, the skew adjacency matrix $A(G^e)$ of an oriented graph G^e has the form

$$A(G^e) = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

where B is called the *skew biadjacency matrix* of G^e. Let A, B, C and D be $n \times n$ matrices with $\det A \neq 0$ and $AC = CA$. It is well-known that $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \left(AD - CB \right)$. Following from $(xI)B^T = B^T(xI)$, we obtain that $\det(xI - A(G^e)) = \det \left(xI - \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right)$

$= \det \begin{pmatrix} xI & -B \\ B^T & xI \end{pmatrix} = \det(x^2 I + B^T B)$. Using the result of Corollary 2.6 and Theorem 4.3, we have the following consequence.

Theorem 4.5. *Let $G = (U, V)$ be a 2-connected bipartite graph with $|U| = |V|$ containing no even subdivision of $K_{2,3}$. Then $\pi(G, x) = \det(x^2 I + B^T B)$, where B is the skew biadjacency matrix of a Pfaffian orientation G^e of G.*

4.3 Examples

We now give some examples to compute the permanental polynomials of bipartite graphs containing no even subdivision of $K_{2,3}$.

Example 1. Let G_1^s be the graph with s pairwise internally disjoint paths of length three joining two given vertices. See Figure 2(a). We can see that G_1^s is a bipartite graph with $|U|=|V|$. By Theorems 3.4 and 3.9 we know that G_1^s contains no even subdivision of $K_{2,3}$.
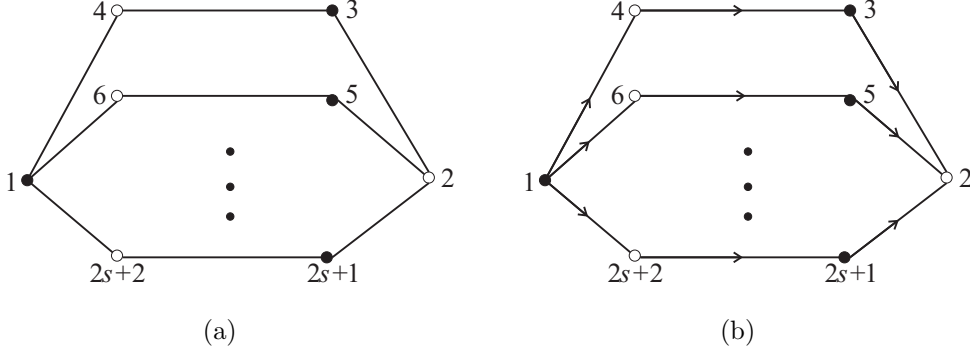

Figure 2. Graph G_1^s and its Pfaffian orientation $(G_1^s)^e$.

Lemma 4.6. $D_n = \det \begin{pmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_n \end{pmatrix} = (1 + b\sum_{i=1}^{n} \frac{1}{a_i - b}) \prod_{i=1}^{n}(a_i - b)$, where $b \neq a_i, i = 1, 2, \cdots, n$.

Proof. $D_n = \det \begin{pmatrix} 1 & b & b & b & \cdots & b \\ 0 & a_1 & b & b & \cdots & b \\ 0 & b & a_2 & b & \cdots & b \\ 0 & b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b & b & b & \cdots & a_n \end{pmatrix} = \det \begin{pmatrix} 1 & b & b & b & \cdots & b \\ -1 & a_1 - b & 0 & 0 & \cdots & 0 \\ -1 & 0 & a_2 - b & 0 & \cdots & 0 \\ -1 & 0 & 0 & a_3 - b & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & a_n - b \end{pmatrix}$

$= \det \begin{pmatrix} 1 + \frac{b}{a_1-b} + \cdots + \frac{b}{a_n-b} & b & b & b & \cdots & b \\ & a_1 - b & & & & \\ & & a_2 - b & & & \mathbf{0} \\ & & & a_3 - b & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & a_n - b \end{pmatrix}$

$= (1 + b\sum_{i=1}^{n} \frac{1}{a_i - b}) \prod_{i=1}^{n}(a_i - b).$ $\qquad\square$

Theorem 4.7. $\pi(G_1^s, x) = (1 + \frac{1}{x^2+s-1} + \frac{s}{x^2+1})(x^2 + s - 1)(x^2 + 1)^s.$

Proof. We choose a Pfaffian orientation $(G_1^s)^e$ so that each of the s paths is oriented as a directed path from vertex 1 to vertex 2 (see Figure 2(b)). Since G is 2-connected and contains no even subdivision of $K_{2,3}$, by Theorem 4.5 we have that $\pi(G_1^s, x) = \det(x^2 I + B^T B)$, where

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$B_{(s+1)\times(s+1)}$ the skew biadjacency matrix of $(G_1^s)^e$. As the labeling of vertices in Figure 2(b), the skew biadjacency matrix B has the following form

$$
B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}, \quad \text{and } B^T B = \begin{pmatrix} s & -1 & -1 & \cdots & -1 \\ -1 & 2 & 1 & \cdots & 1 \\ -1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & 1 & \cdots & 2 \end{pmatrix}.
$$

Hence $\det(x^2 I + B^T B)$

$$
= \det \begin{pmatrix} x^2 + s & -1 & -1 & \cdots & -1 \\ -1 & x^2 + 2 & 1 & \cdots & 1 \\ -1 & 1 & x^2 + 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & 1 & \cdots & x^2 + 2 \end{pmatrix} = \det \begin{pmatrix} x^2 + s & 1 & 1 & \cdots & 1 \\ 1 & x^2 + 2 & 1 & \cdots & 1 \\ 1 & 1 & x^2 + 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & x^2 + 2 \end{pmatrix}
$$

$= (1 + \frac{1}{x^2+s-1} + \frac{s}{x^2+1})(x^2 + s - 1)(x^2 + 1)^s$, by Lemma 4.6. $\qquad\square$

Example 2. Let $G_2^r = (K_{1,r} \times K_2)^*$ be obtained from the Cartesian product $K_{1,r} \times K_2$ by adding paths of length three joining the adjacent vertices u and v of $K_{1,r} \times K_2$ with $u \in V(K_{1,r}^1) - \{x\}$ and $v \in V(K_{1,r}^2) - \{x\}$ ($K_{1,r}^1$ and $K_{1,r}^2$ are the two copies of $K_{1,r}$ in $K_{1,r} \times K_2$ and x is the vertex of degree r in $K_{1,r}$). See Figure 3(a). Similar to Example 1, G_2^r is a bipartite graph with $|U| = |V|$ and contains no even subdivision of $K_{2,3}$.
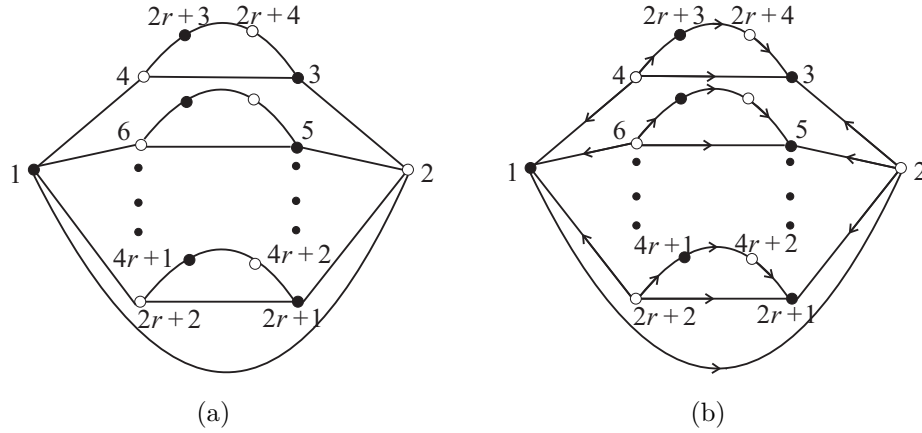

Figure 3.   Graph G_2^r and its Pfaffian orientation $(G_2^r)^e$.

Theorem 4.8. $\pi(G_2^r, x) = (1 + \frac{r}{x^2+2})(x^2 + 2)^{2r-1}[x^4 + (3+r)x^2 + r + 2]$.

Proof. Orientate G_2^r as follows. Direct the edges in the star $K_{1,r}^1$ from the vertices of degree one to the vertex of degree r and the edges in the star $K_{1,r}^2$ receive the reverse direction; Edges joining $K_{1,r}^1$ and $K_{1,r}^2$ are directed from $K_{1,r}^1$ to $K_{1,r}^2$; Each path of length three added to $K_{1,r} \times K_2$ is oriented as a directed path from $K_{1,r}^1$ to $K_{1,r}^2$ (see Figure 3(b)). It can be

easily checked that each face cycle is oddly oriented. Hence $(G_2^r)^e$ is a Pfaffian orientation. Since G_2^r is 2-connected and contains no even subdivision of $K_{2,3}$, from Theorems 4.5 we obtain that $\pi(G_2^r, x) = \det(x^2 I + B^T B)$, where the $(2r+1) \times (2r+1)$ matrix B is the skew biadjacency matrix of $(G_2^r)^e$. By labeling the vertices of $(G_2^r)^e$ as shown in Figure 3(b), we obtain that

$$
B = \begin{pmatrix}
1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\
-1 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
-1 & 0 & -1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & -1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

and

$$
B^T B = \begin{pmatrix}
r+1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 3 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 3 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 1 & \cdots & 3 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 2
\end{pmatrix}.
$$

By the properties of determinants, we have that

$$
\det(x^2 I + B^T B) = \det \begin{pmatrix}
x^2 + (r+1) & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & x^2 + 3 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & x^2 + 3 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 1 & 1 & \cdots & x^2 + 3 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & x^2 + 2 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & x^2 + 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & x^2 + 2
\end{pmatrix}
$$

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$$= \det \begin{pmatrix} x^2+(r+1) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & x^2+3 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & x^2+3 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 1 & \cdots & x^2+3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2(x^2+2) & x^2+2 & \cdots & -x^2-2 \\ 0 & 0 & 0 & \cdots & 0 & x^2+2 & 2(x^2+2) & \cdots & -x^2-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & -x^2-2 & -x^2-2 & \cdots & x^2+2 \end{pmatrix}$$

(Since B and $B^T B$ are of order $2r+1$, the two matrices as above are also of order $2r+1$.)

$$= (x^2+r+1)\det \begin{pmatrix} x^2+3 & 1 & \cdots & 1 \\ 1 & x^2+3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x^2+3 \end{pmatrix} \det \begin{pmatrix} 2(x^2+2) & x^2+2 & \cdots & -x^2-2 \\ x^2+2 & 2(x^2+2) & \cdots & -x^2-2 \\ \vdots & \vdots & \ddots & \vdots \\ -x^2-2 & -x^2-2 & \cdots & x^2+2 \end{pmatrix}$$

$$-\det \begin{pmatrix} x^2+3 & 1 & \cdots & 1 \\ 1 & x^2+3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x^2+3 \end{pmatrix} \det \begin{pmatrix} 2(x^2+2) & x^2+2 & \cdots & x^2+2 \\ x^2+2 & 2(x^2+2) & \cdots & x^2+2 \\ \vdots & \vdots & \ddots & \vdots \\ x^2+2 & x^2+2 & \cdots & 2(x^2+2) \end{pmatrix}$$

(The first three matrices as above are of order r and the last one is of order $r-1$; Lemma 4.6 is used repeatedly)

$$= (x^2+r+1)(1+\tfrac{r}{x^2+2})(x^2+2)^r \det \begin{pmatrix} x^2+2 & 0 & \cdots & -x^2-2 \\ 0 & x^2+2 & \cdots & -x^2-2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^2+2 \end{pmatrix} -r(1+\tfrac{r}{x^2+2})(x^2+2)^{2r-1}$$

$$= (x^2+r+1)(1+\tfrac{r}{x^2+2})(x^2+2)^{2r} - r(1+\tfrac{r}{x^2+2})(x^2+2)^{2r-1}$$

$$= (1+\tfrac{r}{x^2+2})(x^2+2)^{2r-1}[x^4+(3+r)x^2+r+2]. \qquad \square$$

Example 3. We consider outerplanar bipartite graphs. For example, see Figure 4(a). If all the polygons in this graph are hexagons, then the resulting graph is a catacondensed hexagonal system (see Figure 4(b)).

Let H be the graph in Figure 4(b) with an orientation H^e in Figure 4(c) with each cycle oddly oriented. Let $A(H^e)$ be the skew adjacency matrix of H^e. We have that $\pi(H,x) = \det(xI - A(H^e))$ by Theorem 3.11 and Corollary 2.6. After computation we obtain that $\pi(H,x) = 81 + 648x^2 + 2106x^4 + 3627x^6 + 3645x^8 + 2223x^{10} + 825x^{12} + 180x^{14} + 21x^{16} + x^{18}$.

Borowiechi [2] ever proved that if a bipartite graph G contains no cycle of length $4s$, $s \in \{1,2,\cdots\}$, and the characteristic polynomial $\phi(G,x) = \Sigma_{k=0}^{[n/2]}(-1)^k a_{2k} x^{n-2k}$, then $\pi(G,x) =$
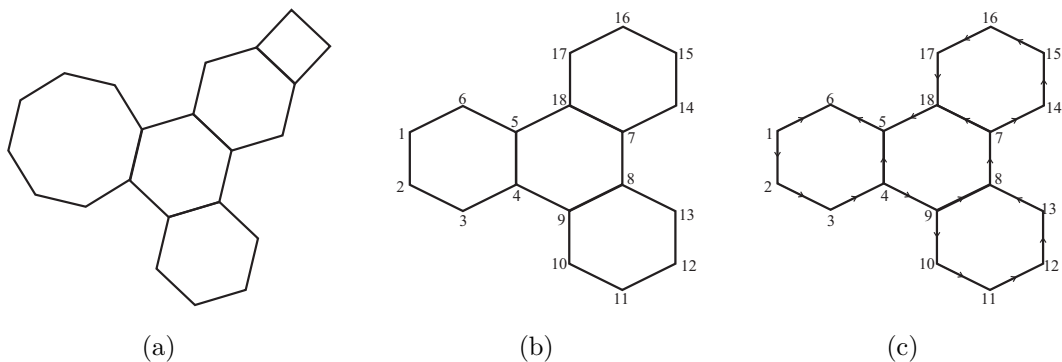
Figure 4. (a) An Outerplanar graph, (b) a catacondensed hexagonal system, and (c) an Pfaffian orientation.

$\Sigma_{k=0}^{[n/2]} a_{2k} x^{n-2k}$. For example, we can compute $\phi(H, x) = -81 + 648x^2 - 2106x^4 + 3627x^6 - 3645x^8 + 2223x^{10} - 825x^{12} + 180x^{14} - 21x^{16} + x^{18}$. Hence the permanental polynomials of the catacondensed hexagonal system can be computed in such two methods, since they contains no cycle of length $4s$.
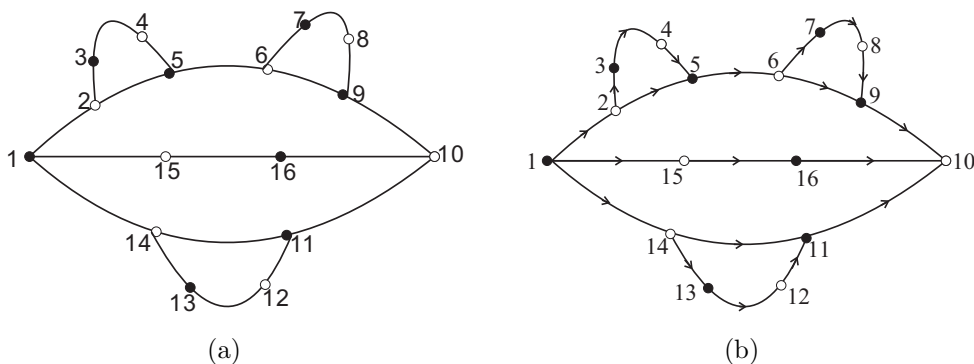

Figure 5. A graph containing no even subdivision of $K_{2,3}$ and its Pfaffian orientation.

Example 4. Figure 5(a) is a bipartite graph G containing no even subdivision of $K_{2,3}$. An orientation G^e obtained by Algorithm 4.1 is given in Figure 5(b). Let $A(G^e)$ be the skew adjacency matrix of G^e. By Corollary 2.6, we have that $\pi(G, x) = \det(xI - A(G^e)) = 196 + 1108x^2 + 2433x^4 + 2780x^6 + 1832x^8 + 718x^{10} + 164x^{12} + 20x^{14} + x^{16}$.

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